

# Thermodynamics of Superstring on Near-extremal NS5 and Effective Hagedorn Behavior

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## Abstract

We study the thermodynamical torus partition function of superstring on the near-extremal black NS5-brane background. The exact partition function has been computed with the helps of our previous works: [arXiv:1012.5721 [hep-th]], [arXiv:1109.3365 [hep-th]], and naturally decomposed into two parts. The first part is contributed from strings freely propagating in the asymptotic region, which are identified as the superstring gas at the Hawking temperature on the linear-dilaton background. The second part includes the contribution localized around the ‘tip of cigar’, which characterizes the non-extremality. Remarkably, the latter part includes massless excitations with non-vanishing thermal winding, which signifies that the Hagedorn-like behavior effectively appears, even though the Hawking temperature is much lower than the Hagedorn temperature. We also explore the high-temperature backgrounds defined by the orbifolding along the Euclidean time direction. In those cases, the thermal winding modes localized around the tip are found to be tachyonic, reflecting the singularities of Euclidean backgrounds caused by orbifolding.

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# 1 Introduction

One of familiar interesting features of black-hole (BH) physics is the emergence of thermodynamics. After making the Wick rotation, the Euclidean geometry is smoothly defined only if the imaginary time axis is asymptotically compactified to a circle with a definite radius, which defines the Hawking temperature. Note that the asymptotic thermal circle in a typical Euclidean BH background (say, Euclidean Schwarzschild BH) is topologically trivial and contractible to a point at the ‘location of horizon’, which we denote  $r = r_0$  here. Now, if considering closed strings on such BH backgrounds, an interesting phenomenon would happen: closed strings wound around the small circle very close to the point  $r = r_0$  could be tachyonic, at least naively, no matter how low value of the Hawking temperature (*i.e.* large asymptotic circle) we assume. The emergence of ‘thermal winding tachyons’ signifies the Hagedorn behavior [1] in thermal string theories [2, 3]. One of the main purposes of this paper is to show that such an ‘effective Hagedorn behavior’ actually arises from the viewpoint of exact world-sheet analysis of closed superstring theory.

We study thermodynamical features of the superstring on near-extremal black NS5-brane background, which shares several physical properties with the Schwarzschild BH. Especially, their Euclidean geometries would resemble at least qualitatively. As first pointed out in [4] based on the black 5-brane solution given in [5], the RNS superstring in the near horizon region of this background is described by an exactly soluble superconformal system;

$$\frac{SL(2, \mathbb{R})}{U(1)\text{-supercoset}} \times SU(2)\text{-superWZW} \times \mathbb{R}^5.$$

Here the  $SL(2, \mathbb{R})/U(1)$ -supercoset model [6] is identified with the (supersymmetric) 2-dimensional black-hole (2DBH) [7]. Utilizing this fact, we study the 1-loop thermal partition function on this background. The most non-trivial part of our analysis lies in the sector of Euclidean 2DBH (‘cigar SCFT’) whose asymptotic circle determines the Hawking temperature of the black NS5-background. We should also carefully work with the contributions of RNS world-sheet fermions with suitable boundary conditions as the thermal superstring theory [3]. With the helps of our previous works [8, 9] we analyze the exact thermal partition function, aiming mainly at understanding of thermal properties of the system.

This paper is organized as follows:

In section 2, as a preliminary, we make a brief review of near-extremal black NS5-brane system and its interpretation as a superconformal system under the near-horizon limit [4].

In section 3, we evaluate the thermal partition function on this background. This analysis is parallel to those given in our previous works [8, 9], but includes some extensions so that spin

structures are included. After making the IR-regularization as given in [8], we can naturally decompose the obtained partition function into two parts;

$$Z^{(\text{reg})}(\tau) = Z^{(\text{asp})}(\tau) + Z^{(\text{fin})}(\tau),$$

that is, the ‘asymptotic part’ and the ‘finite part’. The former is identified as the one proportional to  $-\log \varepsilon$ , where  $\varepsilon$  is the regularization parameter. This factor is interpreted as the divergent volume factor. On the other hand, the latter is finite under the  $\varepsilon \rightarrow +0$  limit. It is interpreted as contributions from the strongly curved region near the tip of cigar.

In section 4, we shall make an analysis of spectra read off from the partition function, mainly focusing on the light excitations. The analysis with respect to the finite part  $Z^{(\text{fin})}(\tau)$  is a main result of this paper. Among other things, we will observe the emergence of the ‘effective Hagedorn behavior’ suggested above. Namely,  $Z^{(\text{fin})}(\tau)$  behaves as if we were just at the Hagedorn temperature, although the Hawking temperature is much lower than the Hagedorn one. We also examine the high temperature backgrounds defined by the  $\mathbb{Z}_M$ -orbifolding along the Euclidean time direction, which extend singularities at the tip. In those cases we will find out thermal behaviors with temperatures exceeding the Hagedorn one.

In section 5, we give a summary and some additional comments.

## 2 Near Extremal Black NS5-brane Background

The supergravity solution for the near extremal black NS5-branes ( $\sharp \text{NS5} = k \in \mathbb{Z}_{>0}$ ) is written as [5]

$$\begin{aligned} ds^2 &= - \left(1 - \frac{r_0^2}{r^2}\right) dt^2 + \left(1 + \frac{k\alpha'}{r^2}\right) \left(\frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2 d\Omega_3^2\right) + dy_5^2, \\ e^{2\Phi} &= g_s^2 \left(1 + \frac{k\alpha'}{r^2}\right), \end{aligned} \quad (2.1)$$

where  $r = r_0$  is the horizon, and  $g_s$  denotes the string coupling at the asymptotic region  $r \rightarrow \infty$ . (The familiar extremal solution corresponds to  $r_0 = 0$ .) It is convenient to introduce the parameter

$$\mu \equiv \frac{r_0^2}{g_s^2 \alpha'}, \quad (2.2)$$

which is identified as the energy density above extremality. The near horizon limit is defined by taking the limit  $r_0, g_s \rightarrow 0$  with  $\mu$  kept finite. Defining new variable  $\rho$  by  $r = r_0 \cosh\left(\frac{\rho}{\sqrt{k\alpha'}}\right)$ ,

we obtain the near horizon geometry [4];

$$ds^2 = -\tanh^2\left(\frac{\rho}{\sqrt{k\alpha'}}\right) dt^2 + d\rho^2 + k\alpha' d\Omega_3^2 + dy_5^2, \quad (2.3)$$

$$e^{2\Phi} = \frac{k}{\mu \cosh^2\left(\frac{\rho}{\sqrt{k\alpha'}}\right)}.$$

The  $(t, \rho)$ -sector is identified with the 2-dimensional black-hole (2DBH) [7], while the  $S^3$  part corresponds to the (super)  $SU(2)$  WZW model together with the implicit Kalb-Ramond field  $B_{\mu\nu}$  in a familiar manner (with the *bosonic* level  $k-2$ ). We also compactify the directions parallel to the NS5s ( $y$ -directions) to a 5-torus  $T^5$ . We have thus found that the type II string on this background is described by the superconformal system;

$$\frac{SL(2, \mathbb{R})_{k+2}}{U(1)} \times SU(2)_{k-2} \times T^5. \quad (2.4)$$

The criticality condition is satisfied as

$$\frac{3(k+2)}{k} + \left(\frac{3(k-2)}{k} + \frac{3}{2}\right) + \frac{3}{2} \times 5 = 15. \quad (2.5)$$

As mentioned in [4], the system is weakly coupled in the both senses of world-sheet and space-time, when

$$k \gg 1, \quad \frac{\mu}{k} \gg 1. \quad (2.6)$$

is satisfied. We shall assume it throughout this paper.

The Wick rotation  $t \rightarrow -i\sqrt{k\alpha'}t_E$  converts the system into a thermal model:

$$ds^2 = k\alpha' \tanh^2\left(\frac{\rho}{\sqrt{k\alpha'}}\right) dt_E^2 + d\rho^2 + k\alpha' d\Omega_3^2 + dy_5^2. \quad (2.7)$$

which amounts to replacing the Lorentzian 2DBH with the Euclidean 2DBH realized as the ‘cigar’ geometry. As is well-known, if requiring the smoothness of geometry, the Euclidean time  $t_E$  need possess the periodicity  $t_E \cong t_E + 2\pi$ , and hence the asymptotic radius of cigar is fixed to be  $\sqrt{\alpha'k}$ . This just means that the present black-hole background has the Hawking temperature:

$$T_{\text{Hw}} \equiv \beta_{\text{Hw}}^{-1}, \quad \beta_{\text{Hw}} = 2\pi\sqrt{\alpha'k}. \quad (2.8)$$

It is familiar that the Hagedorn temperature of free string gas is uniquely determined from the ‘effective central charge’  $c_{\text{eff}}$  [10] of the transverse sector as

$$T_{\text{Hg}} \equiv \beta_{\text{Hg}}^{-1}, \quad \beta_{\text{Hg}} = 2\pi\sqrt{\frac{\alpha'c_{\text{eff}}}{6}}, \quad (2.9)$$

by means of the Cardy formula. In the present background, this leads to

$$\beta_{\text{Hg}} = 2\pi \sqrt{2\alpha' \left(1 - \frac{1}{2k}\right)}, \quad (2.10)$$

because of  $c_{\text{eff}} = 12 - 24 \times \frac{1}{4k} = 12 \left(1 - \frac{1}{2k}\right)$ , where the correction  $\frac{1}{4k}$  is the mass gap that originates from the (asymptotic) linear dilaton term<sup>1</sup>. Since we assumed a sufficiently large  $k(\equiv \# \text{NS5})$ , we have  $T_{\text{Hw}} \ll T_{\text{Hg}}$ .

### 3 Thermal Partition Function

In this section we evaluate the thermal torus partition function with the Hawking temperature (2.8). To this aim, it is convenient to separate the contribution depending on the spin structures, which includes the  $SL(2, \mathbb{R})/U(1)$ -supercoset, 8 free fermions, and also the superconformal ghosts  $\beta, \gamma$ . Namely, the desired partition function is written as ( $\tau \equiv \tau_1 + i\tau_2$ )

$$Z(\tau) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{\sigma_L, \sigma_R : \text{spin structure}} Z_{[\sigma_L, \sigma_R]}(\tau) Z_{SU(2)}(\tau) Z_{T^5}(\tau) Z_{bc\text{-gh}}(\tau), \quad (3.1)$$

where  $Z_{[\sigma_L, \sigma_R]}(\tau)$  denotes the relevant part depending on the spin structures, while  $Z_{SU(2)}(\tau)$  and  $Z_{T^5}(\tau)$  denote the bosonic parts of  $SU(2)$ ,  $T^5$  sectors that are independent of spin structures.  $\mathcal{F}$  denotes the fundamental region as usual;

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} \mid |\tau| \geq 1, \quad -\frac{1}{2} \leq \tau_1 < \frac{1}{2} \right\}, \quad (\mathbb{H} \equiv \{z \in \mathbb{C} \mid \text{Im } z > 0\}).$$

The modular invariance of  $Z(\tau)$  especially requires the ‘modular covariance’ of the spin-structure part  $Z_{[\sigma_L, \sigma_R]}(\tau)$  expressed as

$$Z_{[\sigma_L, \sigma_R]}(\tau + 1) = Z_{[T \cdot \sigma_L, T \cdot \sigma_R]}(\tau), \quad Z_{[\sigma_L, \sigma_R]} \left( -\frac{1}{\tau} \right) = Z_{[S \cdot \sigma_L, S \cdot \sigma_R]}(\tau). \quad (3.2)$$

Here we introduced the notations;

$$T \cdot \text{NS} = \widetilde{\text{NS}}, \quad T \cdot \widetilde{\text{NS}} = \text{NS}, \quad T \cdot \text{R} = \text{R}, \quad T \cdot \widetilde{\text{R}} = \widetilde{\text{R}}, \quad (3.3)$$

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<sup>1</sup>To be more precise, we have to take account of the contributions from the discrete representations to determine the effective central charge  $c_{\text{eff}}$ . However, as was shown *e.g.* in [12], the identity rep. (graviton rep.) decouples from the  $SL(2, \mathbb{R})/U(1)$ -sector and the conformal weights of normalizable discrete states are greater than  $\frac{1}{2k}$ . Thus,  $c_{\text{eff}}$  is unchanged even if taking account of this contribution. In order to avoid a possible confusion, we also emphasize that the Hagedorn temperature  $T_{\text{Hg}}$  considered through this paper is defined with respect to the fundamental superstring, not to the so-called ‘Little String Theory’ (LST) [11]. See also the second comment given at the end of section 5.

$$S \cdot \text{NS} = \text{NS}, \quad S \cdot \widetilde{\text{NS}} = \text{R}, \quad S \cdot \text{R} = \widetilde{\text{NS}}, \quad S \cdot \widetilde{\text{R}} = \widetilde{\text{R}}. \quad (3.4)$$

Now, let us focus on  $Z_{[\sigma_L, \sigma_R]}(\tau)$ . According to the analysis [12, 8, 9] on the  $SL(2, \mathbb{R})/U(1)$ -supercoset, it is written in the form of

$$\begin{aligned} Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) &= k \sum_{m_i \in \mathbb{Z}} \int_{\varepsilon}^{1-\varepsilon} ds_1 \int_0^1 ds_2 \epsilon(\sigma_L; m_1, m_2) \epsilon(\sigma_R; m_1, m_2) \\ &\quad \times f_{[\sigma_L]}(\tau, s_1\tau + s_2) [f_{[\sigma_R]}(\tau, s_1\tau + s_2)]^* e^{-\frac{\pi k}{\tau_2} |(s_1+m_1)\tau + (s_2+m_2)|^2} \end{aligned} \quad (3.5)$$

where we set

$$f_{[\sigma]}(\tau, u) := \frac{\theta_{[\sigma]}(\tau, u)}{\theta_1(\tau, u)} \left( \frac{\theta_{[\sigma]}(\tau, 0)}{\eta(\tau)} \right)^3, \quad (3.6)$$

with the abbreviated notation for theta functions (C.1). The factor  $\left(\frac{\theta_{[\sigma]}}{\eta}\right)^3$  is just identified as the contribution from free fermions (and the superconformal ghosts), while  $\frac{\theta_{[\sigma]}(u)}{\theta_1(u)}$  originates from the  $SL(2, \mathbb{R})/U(1)$ -part [12, 8, 9]. The integers  $m_1, m_2$  are identified with the winding numbers around the (asymptotic) thermal circle with the radius  $\frac{\beta_{\text{Hw}}}{2\pi} = \sqrt{\alpha' k}$ , that is,

$$m_1 = \text{spatial winding}, \quad m_2 = \text{temporal winding}.$$

Moreover, we introduced the phase factor<sup>2</sup>

$$\epsilon(\sigma; m_1, m_2) \equiv \begin{cases} 1 & (\sigma = \text{NS}) \\ (-1)^{m_1+1} & (\sigma = \widetilde{\text{NS}}) \\ (-1)^{m_2+1} & (\sigma = \text{R}) \\ (-1)^{m_1+m_2} & (\sigma = \widetilde{\text{R}}) \end{cases} \quad (3.7)$$

to include the correct boundary condition for world-sheet fermions in the thermal superstring [3]. The IR regularization as given in [8] has been made with a positive small parameter  $\varepsilon$ .

To proceed further, we shall introduce the integer parameters  $N$  and  $K$  such that  $k = N/K$  as in [8, 9] and *assume*  $K \in 2\mathbb{Z}_{>0}$ <sup>3</sup>. One can evaluate the integrations of moduli  $s_1, s_2$  in a way parallel to [8]. We thus just sketch how to evaluate it with making emphasis about differences from [8];

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<sup>2</sup>Of course, the definition for  $\sigma = \widetilde{\text{R}}$  is just formal, since this sector does not contribute to the partition function. (3.7) here corresponds to the type IIB string.

<sup>3</sup>We do not assume  $N$  and  $K$  are coprime, and thus this assumption can be always satisfied. Although  $k$  is a positive integer here (since it is identified with the NS5 charge), *we shall not choose*  $K = 1$ . It might sound unnatural, but this is a convenient assumption for our analysis of the ‘character decomposition’ of the partition function with the *every* spin structure. It is found that all the results we give in this paper do not depend on the choice of  $N$  and  $K$  as long as the condition  $K \in 2\mathbb{Z}_{>0}$  is satisfied. Of course, the simplest choice would be  $N = 2k$  and  $K = 2$ .

- The temporal winding  $m_2$  is dualized into the KK momentum  $n$  by means of Poisson resummation. When performing it, one must correctly take account of the extra phase factors depending on the spin structures. For instance, we simply obtain

$$\sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi k}{\tau_2} \{(s_1+m_2)\tau_1 + (s_2+m_2)\}^2} = \sqrt{\frac{\tau_2}{k}} \sum_{n \in \mathbb{Z}} e^{-\pi \tau_2 \frac{n^2}{k} + 2\pi i n \{(s_1+m_2)\tau_1 + s_2\}},$$

for the NS-NS sector. However, the extra phase (3.7) yields  $n \in \mathbb{Z} + \frac{1}{2}$  *e.g.* in the NS-R sector.

- To make the integral over the  $U(1)$ -modulus  $u \equiv s_1\tau + s_2$ , it is convenient to utilize the identities (A.5) in the factor (3.6). For example, we obtain

$$f_{[\text{NS}]}(\tau, u) = \left( \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right)^3 \frac{\theta_3(\tau, 0)}{i\eta(\tau)^3} \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} \frac{e^{2\pi i u \nu}}{1 + q^\nu}, \quad (3.8)$$

for the NS-sector. Here, the powers of  $q$ -expansion  $\nu$  take values in half-integers for the NS and  $\widetilde{\text{NS}}$  sectors, and integers for the R (and  $\widetilde{\text{R}}$ ) sector. Note also that the contribution from  $\widetilde{\text{R}}$ -sector trivially vanishes. The  $s_2$ -integral imposes the constraints

$$\nu - \tilde{\nu} = n, \quad (3.9)$$

where  $\nu, \tilde{\nu}$  denote the powers of  $q$ -expansions in the left and right movers, respectively. This constraint is always meaningful for any spin structure. For example, in the case of NS-R sector, both sides of (3.9) take values in half-integers due to the above remark.

- It is convenient to introduce a combined quantum number  $v$  defined by

$$v := Nm_1 - K(\nu + \tilde{\nu}), \quad (3.10)$$

and to express the partition function in terms of a summation over  $\nu, \tilde{\nu}$  and  $v$  with a constraint  $v + K(\nu + \tilde{\nu}) \in N\mathbb{Z}$ . Here  $v$  is always an integer, since we are assuming  $K \in 2\mathbb{Z}_{>0}$ . We also remark that the partition function gets a phase  $(-1)^{\frac{n+K(\nu+\tilde{\nu})}{N}}$  for the NS- $\widetilde{\text{NS}}$ ,  $\widetilde{\text{NS}}$ -NS, R- $\widetilde{\text{NS}}$ , and  $\widetilde{\text{NS}}$ -R sectors, which originates from (3.7).

- We make use of the following identity to carry out the  $s_1$ -integral as in [8] (see also [13]);

$$\sqrt{k\tau_2} \int_{\varepsilon}^{1-\varepsilon} ds_1 e^{-\pi \tau_2 \frac{N}{K} s_1^2 - 2\pi \tau_2 s_1 \frac{v}{K}} = \frac{1}{2\pi i} \int_{\mathbb{R}-i0} dp \frac{e^{-\pi \tau_2 \frac{p^2}{NK}}}{p - iv} \left\{ e^{-2\pi i \varepsilon \frac{\tau_2}{K} (p-iv)} - e^{-2\pi i (1-\varepsilon) \frac{\tau_2}{K} (p-iv)} \right\}. \quad (3.11)$$

Combining all the pieces, we finally obtain

$$\begin{aligned}
Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) &= \sum_{v \in \mathbb{Z}} \sum_{\substack{\nu \in \mathbb{Z} + \frac{s(\sigma_L)-1}{2}, \tilde{\nu} \in \mathbb{Z} + \frac{s(\sigma_R)-1}{2} \\ v+K(\nu+\tilde{\nu}) \in N\mathbb{Z}}} \epsilon(\sigma_L; v, \nu, \tilde{\nu}) \epsilon(\sigma_R; v, \nu, \tilde{\nu}) \\
&\times \frac{1}{2\pi i} \left[ \int_{\mathbb{R}-i0} dp q^\nu \bar{q}^{\tilde{\nu}} (-1)^{t(\sigma_L)+t(\sigma_R)} e^{-\varepsilon'(v+ip)} - \int_{\mathbb{R}+i(N-0)} dp e^{\varepsilon'(v+ip)} \right] \\
&\times \frac{e^{-\pi\tau_2 \frac{p^2+v^2}{NK}}}{p-iv} \frac{q^{\frac{K}{N}\nu^2 + \frac{v}{N}\nu}}{1+(-1)^{t(\sigma_L)}q^\nu} \frac{\overline{q^{\frac{K}{N}\tilde{\nu}^2 + \frac{v}{N}\tilde{\nu}}}}{1+(-1)^{t(\sigma_R)}q^{\tilde{\nu}}} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^4 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^4} \left| \frac{1}{\eta^2} \right|^2, \quad (3.12)
\end{aligned}$$

where we set  $\varepsilon' := 2\pi \frac{\tau_2}{K} \varepsilon$ , and the phase factor  $\epsilon(\sigma; v, \nu, \tilde{\nu})$  is defined by

$$\epsilon(\sigma; v, \nu, \tilde{\nu}) := \begin{cases} 1 & (\sigma = \text{NS}) \\ (-1)^{\frac{v+K(\nu+\tilde{\nu})}{N}+1} & (\sigma = \widetilde{\text{NS}}) \\ -1 & (\sigma = \text{R}) \\ (-1)^{\frac{v+K(\nu+\tilde{\nu})}{N}} & (\sigma = \widetilde{\text{R}}) \end{cases} \quad (3.13)$$

We also introduced the notation

$$s(\sigma) := \begin{cases} 0 & \sigma = \text{NS}, \widetilde{\text{NS}}, \\ 1 & \sigma = \text{R}, \widetilde{\text{R}} \end{cases} \quad t(\sigma) := \begin{cases} 0 & \sigma = \text{NS}, \text{R}, \\ 1 & \sigma = \widetilde{\text{NS}}, \widetilde{\text{R}} \end{cases} \quad (3.14)$$

The regularized partition function  $Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon)$  correctly behaves under the modular T-transformation;

$$Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau + 1; \varepsilon) = Z_{[T \cdot \sigma_L, T \cdot \sigma_R]}^{(\text{reg})}(\tau; \varepsilon), \quad (3.15)$$

where we used the notation (3.3). On the other hand, the modular covariance for the S-transformation is ‘weakly’ violated;

$$Z_{[\sigma_L, \sigma_R]}^{(\text{reg})} \left( -\frac{1}{\tau}; \varepsilon \right) = Z_{[S \cdot \sigma_L, S \cdot \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) + O(\varepsilon, \varepsilon \log \varepsilon), \quad (3.16)$$

with the notation (3.4). The S-transformation relation (3.16) is most easily shown by the definition (3.5) itself. In fact, if rewriting (3.5) as

$$Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) \equiv \int_{\varepsilon}^{1-\varepsilon} ds_1 \int_0^1 ds_2 F_{[\sigma_L, \sigma_R]}(s_1, s_2; \tau),$$

and by using the modular property

$$F_{[\sigma_L, \sigma_R]} \left( s_1, s_2; -\frac{1}{\tau} \right) = F_{[S \cdot \sigma_L, S \cdot \sigma_R]}(s_2, -s_1; \tau) \equiv F_{[S \cdot \sigma_L, S \cdot \sigma_R]}(s_2, 1-s_1; \tau),$$



we obtain

$$\begin{aligned}
& Z_{[\sigma_L, \sigma_R]}^{(\text{reg})} \left( -\frac{1}{\tau}; \varepsilon \right) - Z_{[S \cdot \sigma_L, S \cdot \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) \\
&= \left[ \int_0^\varepsilon ds_1 \int_\varepsilon^{1-\varepsilon} ds_2 + \int_{1-\varepsilon}^1 ds_1 \int_\varepsilon^{1-\varepsilon} ds_2 - \int_\varepsilon^{1-\varepsilon} ds_1 \int_0^\varepsilon ds_2 - \int_\varepsilon^{1-\varepsilon} ds_1 \int_{1-\varepsilon}^1 ds_2 \right] \\
&\quad \times F_{[S \cdot \sigma_L, S \cdot \sigma_R]}(s_1, s_2; \tau).
\end{aligned}$$

This integral over the moduli  $s_i$  at most behaves as  $\sim \varepsilon \log \varepsilon$  under  $\varepsilon \rightarrow +0$ .

## 4 Analysis of Spectra

In this section we study the spectra read off from the partition function, mainly focusing on the light excitations. We shall start with decomposing the partition function (3.12).

### 4.1 Decomposition of Partition Function

As performed in [13, 12, 8], we make use of the manipulation of contour deformation;

$$\int_{\mathbb{R}+i(N-0)} dp[\cdots] = \int_{\mathbb{R}-i0} dp[\cdots] - 2\pi i [\text{residues of poles in } 0 \leq \text{Im } p < N].$$

in order to decompose the partition function (3.12). This yields

$$Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) = Z_{0, [\sigma_L, \sigma_R]}^{(\text{dis})}(\tau) + Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon), \quad (4.1)$$

where the first term is the pole part that is free from the regularization parameter  $\varepsilon$ . One can explicitly write it as

$$\begin{aligned}
Z_{0, [\sigma_L, \sigma_R]}^{(\text{dis})}(\tau) &= \sum_{v=0}^{N-1} \sum_{\substack{\nu \in \mathbb{Z} + \frac{s(\sigma_L)-1}{2}, \tilde{\nu} \in \mathbb{Z} + \frac{s(\sigma_R)-1}{2} \\ v+K(\nu+\tilde{\nu}) \in N\mathbb{Z}}} \epsilon(\sigma_L; v, \nu, \tilde{\nu}) \epsilon(\sigma_R; v, \nu, \tilde{\nu}) \\
&\quad \times \frac{q^{\frac{K}{N}\nu^2 + \frac{v}{N}\nu}}{1 + (-1)^{t(\sigma_L)} q^\nu} \overline{\frac{q^{\frac{K}{N}\tilde{\nu}^2 + \frac{v}{N}\tilde{\nu}}}{1 + (-1)^{t(\sigma_R)} q^{\tilde{\nu}}}} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^4 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^4} \left| \frac{1}{\eta^2} \right|^2 \\
&= \sum_{v=0}^{N-1} \sum_{\substack{a \in \mathbb{Z}_N + \frac{s(\sigma_L)-1}{2}, \tilde{a} \in \mathbb{Z}_N + \frac{s(\sigma_R)-1}{2} \\ v+K(a+\tilde{a}) \in N\mathbb{Z}}} \epsilon(\sigma_L; v, a, \tilde{a}) \epsilon(\sigma_R; v, a, \tilde{a}) \\
&\quad \times \chi_{\text{dis}}^{(\sigma_L)}(v, a; \tau) \overline{\chi_{\text{dis}}^{(\sigma_R)}(v, \tilde{a}; \tau)} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^3 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^3}. \quad (4.2)
\end{aligned}$$

Here  $\chi_{\text{dis}}^{(\sigma)}(v, a; \tau)$  denotes the extended discrete character with the spin structure  $\sigma$  (C.18). Also, the second term is written as

$$\begin{aligned}
Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon) &= \sum_{v \in \mathbb{Z}} \sum_{\substack{\nu \in \mathbb{Z} + \frac{s(\sigma_L)-1}{2}, \tilde{\nu} \in \mathbb{Z} + \frac{s(\sigma_R)-1}{2} \\ v+K(\nu+\tilde{\nu}) \in N\mathbb{Z}}} \epsilon(\sigma_L; v, \nu, \tilde{\nu}) \epsilon(\sigma_R; v, \nu, \tilde{\nu}) \\
&\times \frac{1}{2\pi i} \int_{\mathbb{R}-i0} dp \frac{e^{-\pi\tau_2 \frac{p^2}{NK}}}{p-iv} q^{\frac{K}{N}(\nu+\frac{v}{2K})^2} \overline{q^{\frac{K}{N}(\tilde{\nu}+\frac{v}{2K})^2}} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^4 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^4} \left| \frac{1}{\eta^2} \right|^2, \\
&\times \left[ \frac{1}{1+(-1)^{t(\sigma_L)} q^{-\nu}} e^{-\varepsilon'(v+ip)} - \frac{1}{1+(-1)^{t(\sigma_R)} q^{\tilde{\nu}}} e^{\varepsilon'(v+ip)} \right. \\
&\quad \left. + \frac{e^{\varepsilon'(v+ip)} - e^{-\varepsilon'(v+ip)}}{\{1+(-1)^{t(\sigma_L)} q^{-\nu}\} \{1+(-1)^{t(\sigma_R)} q^{\tilde{\nu}}\}} \right]. \quad (4.3)
\end{aligned}$$

This part depends on the regularization parameter  $\varepsilon$  and shows a logarithmically divergence under the  $\varepsilon \rightarrow +0$  limit.

**Note:** Despite the ‘natural form’ from the viewpoint of superconformal algebra, (4.2) does not show the simple modular covariance. From this reason it would be rather natural to adopt another decomposition as is discussed in [8];

$$Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) = Z_{[\sigma_L, \sigma_R]}^{(\text{dis})}(\tau) + Z_{[\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon), \quad (4.4)$$

where  $Z_{[\sigma_L, \sigma_R]}^{(\text{dis})}(\tau)$  is defined by replacing  $\chi_{\text{dis}}^{(\sigma)}(v, a; \tau)$  with its modular completion  $\widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a; \tau)$  (C.28) in the R.H.S of (4.2);

$$\begin{aligned}
Z_{[\sigma_L, \sigma_R]}^{(\text{dis})}(\tau) &= \sum_{v=0}^{N-1} \sum_{\substack{a \in \mathbb{Z}_N + \frac{s(\sigma_L)-1}{2}, \tilde{a} \in \mathbb{Z}_N + \frac{s(\sigma_R)-1}{2} \\ v+K(a+\tilde{a}) \in N\mathbb{Z}}} \epsilon(\sigma_L; v, a, \tilde{a}) \epsilon(\sigma_R; v, a, \tilde{a}) \\
&\times \widehat{\chi}_{\text{dis}}^{(\sigma_L)}(v, a; \tau) \overline{\widehat{\chi}_{\text{dis}}^{(\sigma_R)}(v, \tilde{a}; \tau)} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^3 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^3}, \quad (4.5)
\end{aligned}$$

and precisely satisfies the modular covariance relations;

$$Z_{[\sigma_L, \sigma_R]}^{(\text{dis})}(\tau+1) = Z_{[T \cdot \sigma_L, T \cdot \sigma_R]}^{(\text{dis})}(\tau), \quad Z_{[\sigma_L, \sigma_R]}^{(\text{dis})}\left(-\frac{1}{\tau}\right) = Z_{[S \cdot \sigma_L, S \cdot \sigma_R]}^{(\text{dis})}(\tau). \quad (4.6)$$

It is also worthwhile to remark that the remaining function  $Z_{[\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon)$  is expressible as a sesquilinear form of only the extended continuous characters  $\chi_{\text{con}}^{(\sigma)}(p, m; \tau)$  (C.14) contrary to  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon)$  (4.3), although we shall omit its explicit form here. However, we shall make use of the ‘old decomposition’ (4.1) for the time being, since it seems relatively easier to analyze the mass spectra of excitations based on it.

## 4.2 Asymptotic Part

We first extract the ‘asymptotic part’ of partition function that shows a logarithmic divergence. Namely, we set

$$\begin{aligned}\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau) &:= -\lim_{\varepsilon \rightarrow +0} \left[ \varepsilon \frac{\partial}{\partial \varepsilon} Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) \right] \\ &\equiv -\lim_{\varepsilon \rightarrow +0} \left[ \varepsilon \frac{\partial}{\partial \varepsilon} Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau; \varepsilon) \right],\end{aligned}\quad (4.7)$$

and

$$\begin{aligned}Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon) &= Z_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau; \varepsilon) + Z_{[\sigma_L, \sigma_R]}^{(\text{fin})}(\tau) + O(\varepsilon, \varepsilon \log \varepsilon) \\ &\equiv -\log \varepsilon \widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau) + Z_{[\sigma_L, \sigma_R]}^{(\text{fin})}(\tau) + O(\varepsilon, \varepsilon \log \varepsilon).\end{aligned}\quad (4.8)$$

Here the ‘finite part’  $Z_{[\sigma_L, \sigma_R]}^{(\text{fin})}(\tau)$  is uniquely determined in this decomposition by requiring the independence of the regularization parameter  $\varepsilon$ . We will later examine this part.

Based on (4.3) and (4.7) it is easy to evaluate the explicit form of  $\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau)$ ;

$$\begin{aligned}\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau) &= \frac{1}{\pi} \int_{-\infty}^{\infty} dp \sum_{\substack{w \in \mathbb{Z} \\ n \in \mathbb{Z} + \frac{s(\sigma_L) + s(\sigma_R)}{2}}} \epsilon(\sigma_L; w, 0) \epsilon(\sigma_R; w, 0) \\ &\quad \times q^{\frac{p^2}{4NK} + \frac{(Nw + Kn)^2}{4NK}} \overline{q^{\frac{p^2}{4NK} + \frac{(Nw - Kn)^2}{4NK}}} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^4 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^4} \left| \frac{1}{\eta^2} \right|^2 \\ &= \frac{2}{\pi} \int_0^{\infty} dp \sum_{\substack{w_0 \in \mathbb{Z}_{2K} \\ n_0 \in \mathbb{Z}_N + \frac{s(\sigma_L) + s(\sigma_R)}{2}}} \epsilon(\sigma_L; w_0, 0) \epsilon(\sigma_R; w_0, 0) \\ &\quad \times \chi_{\text{con}}^{(\sigma_L)}(p, Nw_0 + Kn_0; \tau, 0) \overline{\chi_{\text{con}}^{(\sigma_R)}(p, Nw_0 - Kn_0; \tau, 0)} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^3 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^3},\end{aligned}\quad (4.9)$$

where  $\chi_{\text{con}}^{(\sigma)}(p, m; \tau)$  denotes the extended continuous character (C.14) written explicitly as

$$\chi_{\text{con}}^{(\sigma)}(p, m; \tau) \equiv q^{\frac{p^2}{4NK}} \Theta_{m, NK}(\tau, 0) \frac{\theta_{[\sigma]}(\tau, 0)}{\eta(\tau)^3}, \quad (4.10)$$

in terms of the theta function. The thermal winding number  $m_1$  has been rewritten as  $w$  in the first line of (4.9). This partition function essentially coincides with the one studied in [14].

As is directly confirmed from (4.9),  $\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau)$  correctly behaves under modular transformations:

$$\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau + 1) = \widehat{Z}_{[T \cdot \sigma_L, T \cdot \sigma_R]}^{(\text{asp})}(\tau), \quad \widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}\left(-\frac{1}{\tau}\right) = \widehat{Z}_{[S \cdot \sigma_L, S \cdot \sigma_R]}^{(\text{asp})}(\tau). \quad (4.11)$$

The physical interpretation of the asymptotic part is obvious: this sector corresponds to strings freely propagating in the region away from the NS5-branes. After combining other sectors independent of the spin structures,  $\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau)$  is identified with the thermal partition function of the type IIB superstring [3] on the background

$$S_{\beta_{\text{Hw}}}^1 \times \mathbb{R}_\phi \times SU(2)_k \times T^5,$$

where  $S_{\beta_{\text{Hw}}}^1$  denotes the thermal circle with the inverse Hawking temperature (2.8), and  $\mathbb{R}_\phi$  expresses a linear dilaton background with the dilaton gradient  $\mathcal{Q} \equiv \sqrt{\frac{2}{k}}$ . The divergent factor  $-\log \varepsilon$  originates from the infinite volume of asymptotic region. This sector does not give rise to a thermal instability, since the Hawking temperature (2.8) is lower than the Hagedorn one (2.10). (Recall that we assumed a sufficiently large value of  $k$ .)

The asymptotic sector is correctly GSO projected and preserves the space-time SUSY, when going back to the physical background with Lorentzian signature. In other words, the sector with no thermal winding is GSO projected, as is easily confirmed by observing the  $w = 0$  terms in (4.9).

The spectrum of light excitations in the asymptotic sector is summarized as follows:

**(i) no winding spectrum:**

All the states with no thermal winding are massive: the minimal conformal weight is equal  $h = \tilde{h} = \frac{1}{2} + \frac{1}{4k}$ , which is determined by the GSO projection and the linear dilaton.

**(ii) thermal winding spectrum:**

The lightest winding states are the NS-NS states with  $w = \pm 1$ , whose conformal weights are equal  $h = \tilde{h} = \frac{k}{4}$ . This fact is consistent with the value of Hawking temperature (2.8).

### 4.3 Finite Part and the Effective Hagedorn Behavior

Let us next examine the finite part  $Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau)$ , which shows more intriguing features. By the definition (4.8) one may explicitly write

$$Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau) = \lim_{\varepsilon \rightarrow +0} \left[ 1 - (\log \varepsilon) \varepsilon \frac{\partial}{\partial \varepsilon} \right] Z_{[\sigma_L, \sigma_R]}^{(\text{reg})}(\tau; \varepsilon). \quad (4.12)$$

We first note that  $Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau)$  is also modular covariant;

$$Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau + 1) = Z_{[T \cdot \sigma_L, T \cdot \sigma_R]}^{(\text{fn})}(\tau), \quad Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}\left(-\frac{1}{\tau}\right) = Z_{[S \cdot \sigma_L, S \cdot \sigma_R]}^{(\text{fn})}(\tau), \quad (4.13)$$

even though it would appear hard to check it directly. This fact results from the modular properties (3.15), (3.16) as well as (4.11).

This sector is physically interpreted as the contribution from strings localized near the NS5-branes, which are far from free. We now investigate the IR behavior and the spectra of light excitations read from the partition function  $Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau)$ .

### (i) no winding spectrum **F**

Thanks to the good modular behavior (4.13), one can reinterpret the thermal partition function  $\sum_{\sigma_L, \sigma_R} Z_{[\sigma_L, \sigma_R]}^{(\text{fn})}(\tau)$  as a free energy of ‘superstring gas’ localized near the NS5-branes, after combining the  $SU(2)$  and  $T^5$  sectors. This is simply achieved by dropping off the thermal winding number  $m_1$  in (3.12), and by replacing the fundamental region  $\mathcal{F}$  of torus modulus  $\tau$  with the strip region [15]

$$\mathcal{S} := \left\{ \tau \in \mathbb{H} \mid -\frac{1}{2} \leq \tau_1 < \frac{1}{2} \right\}. \quad (4.14)$$

We are especially interested in the IR-behavior of the free energy of localized components, dominated by light excitations with no winding  $m_1 = 0$ . Since we have contributions both from  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{dis})}(\tau)$  (4.2) and  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau)$  (4.3), we shall separately analyze each part:

- $Z_{0, [\sigma_L, \sigma_R]}^{(\text{dis})}(\tau)$ -**part** :

We first point out that this sector is not supersymmetric, even if working on the non-thermal background. Of course, it is not surprising since we are working with the non-BPS NS5-background. In other words, the GSO projection does *not* necessarily act in the usual form *even for the states with no thermal winding* in a sharp contrast with the thermal superstring in flat backgrounds [3]. In an algebraic footing, this originates from the different behaviors of  $\chi_{\text{con}}^{(\sigma)}$  and  $\chi_{\text{dis}}^{(\sigma)}$  under changing the spin structures. The latter contains extra factors such as  $\frac{1}{1+(-1)^{t(\sigma)}q^{\nu}}$  (see (C.18)), which would give rise to a relative sign difference of vacuum states between the NS and  $\widetilde{\text{NS}}$  sectors.

Now, let us explore the lightest states. We treat each spin structure separately:

#### [NS-NS sector]

We note that setting  $m_1 = 0$  amounts to imposing  $v + K(\nu + \tilde{\nu}) = 0$ , (see (3.10)). Therefore, we have to look for leading terms in (4.2) which lie in the spectral flow

orbits (in the discrete extended character  $\chi_{\text{dis}}^{(\text{NS})}$ , in other words) with the constraint:

$$v + K(a + \tilde{a}) = 0, \quad v = 0, \dots, N-1, \quad a, \tilde{a} \in \frac{1}{2} + \mathbb{Z}_N. \quad (4.15)$$

imposed. The lightest excitation is found to be the vacuum state of discrete character  $\chi_{\text{dis}}^{(\text{NS})}$  with

$$v = K, \quad a = \tilde{a} \equiv -\frac{1}{2} \pmod{N}, \quad (4.16)$$

which corresponds to an anti-chiral primary of the  $SL(2, \mathbb{R})/U(1)$ -sector in both of left and right movers. This state is ‘wrong GSO’ projected (NS and  $\widetilde{\text{NS}}$  appear with the same sign. See the above comment.), and possesses conformal weights  $h = \tilde{h} = \frac{1}{2}$ . This means that it is a massless state after combining it with the identity states of  $SU(2)$  and  $T^5$  sectors.

Other candidates of light excitations would be  $a = \frac{1}{2}, \tilde{a} = -\frac{1}{2}$  or  $a = -\frac{1}{2}, \tilde{a} = \frac{1}{2}$  and  $v = 0$ . However, taking account of the correct/wrong GSO projections (*i.e.* the relative sign of vacuum states in the NS and  $\widetilde{\text{NS}}$  representations), one can find out that both has  $h = \tilde{h} = \frac{1}{2} + \frac{1}{2k}$ . They are hence massive states.

### [R-R sector ]

It is obvious that the lightest state corresponds to a Ramond vacuum in  $SL(2, \mathbb{R})/U(1)$ -sector. There is a unique RR-vacuum in the spectral flow orbit of  $v = 0$  and  $a = \tilde{a} = 0 \in \mathbb{Z}_N$  that satisfies the constraints:

$$v + K(a + \tilde{a}) = 0, \quad v = 0, \dots, N-1, \quad a, \tilde{a} \in \mathbb{Z}_N. \quad (4.17)$$

Combining it with the Ramond vacua of free fermions, we obtain the states with conformal weight  $h = \tilde{h} = \frac{1}{2} + \frac{1}{4k} > \frac{1}{2}$ , which are massive excitations.

### [NS-R (R-NS) sector ]

Again, it is obvious that the right mover should correspond to a Ramond vacuum in the  $SL(2, \mathbb{R})/U(1)$ -sector, and the lightest state has the conformal weights  $h = \tilde{h} = \frac{1}{2} + \frac{1}{4k}$  due to the level-matching condition. It is explicitly realized by setting<sup>4</sup>  $v = \frac{K}{2}$ ,  $a = -\frac{1}{2}$  and  $\tilde{a} = 0$ , which satisfies

$$v + K(a + \tilde{a}) = 0, \quad v = 0, \dots, N-1, \quad a \in \frac{1}{2} + \mathbb{Z}_N, \quad \tilde{a} \in \mathbb{Z}_N. \quad (4.18)$$

---

<sup>4</sup>Recall that we assumed  $K \in 2\mathbb{Z}_{\geq 0}$ . Thus, such  $v$  is always an integer.

•  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}(\tau)$ -part :

We have to be a little careful in order to evaluate the contribution of this sector. All the NS-NS states are again not necessarily GSO projected due to the factors  $\frac{1}{1+(-1)^{t(\sigma_L)}q^{-\nu}}$ ,  $\left[\frac{1}{1+(-1)^{t(\sigma_R)}q^{\tilde{\nu}}}\right]^*$  appearing in (4.3). However, these wrong GSO terms have the minimal conformal weight equal to  $\frac{1}{2} + \frac{1}{4k}$ . In fact, if focusing on primary states, we find that the leading terms with the wrong GSO projection always appear in the form of  $(-1)^{t(\sigma_L)}q^{|\nu|}$  or  $(-1)^{t(\sigma_R)}[q^{|\tilde{\nu}|}]^*$  and  $|\nu|, |\tilde{\nu}| \geq \frac{1}{2}$  holds.

Moreover, all of the correctly GSO projected NS-NS states as well as the R-R, NS-R (R-NS) states are massive since they obviously satisfy

$$h, \tilde{h} > \frac{1}{2} + \frac{1}{4k},$$

as in  $\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau)$ .

In conclusion, the no winding spectrum read off from  $Z_{[\sigma_L, \sigma_R]}^{(\text{fin})}(\tau)$  includes a unique massless state and no tachyon in the NS-NS sector, whereas all the states with other spin structures are massive (lying above the mass gap). This result is quite expected: we have a unique NS-NS modulus with no winding that should be identified with the parameter of deviation from extremality  $\mu$  (2.2). The absence of corresponding R-R and NS-R moduli implies that it corresponds to the marginal deformation breaking the space-time SUSY. We also note that all the familiar moduli of relative distances among NS5-branes appearing in the BPS solution should be lifted up, and such deformations are absent in our near-extremal background (2.1). This fact is again consistent with our analysis of spectrum.

**(ii) thermal winding spectrum - ‘effective Hagedorn behavior’F**

We next discuss the thermal winding spectrum which captures the thermodynamical feature of the finite part  $Z^{(\text{fin})}(\tau)$  according to the standard treatment of thermal string theory [2, 3]. In other words, one will be aware of the UV behavior of the free energy of superstring gas by observing the spectrum with non-vanishing winding number  $m_1 \neq 0$  with the help of modular transformation, even though the winding states themselves are not regarded as physical ones.

For our purpose it is enough to analyze the sector with winding number  $m_1 = 1$ , that is, with the constraint:

$$v + K(\nu + \tilde{\nu}) = N, \tag{4.19}$$

since it obviously yields the leading contribution.

Similarly to the analysis of no winding spectrum, one can confirm that all the states satisfying (4.19) in  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{con})}$  have conformal weights greater than  $\frac{k}{4}$ , although a part of terms are again wrong GSO-projected. It is quite anticipated and leads to the same thermal behavior as that of  $\widehat{Z}_{[\sigma_L, \sigma_R]}^{(\text{asp})}(\tau)$  (4.9) consistent with the Hawking temperature (2.8).

However,  $Z_{0, [\sigma_L, \sigma_R]}^{(\text{dis})}$  yields more interesting aspects. We can again find a unique massless winding state in the NS-NS sector. In fact, it is sufficient to look for the lightest vacuum state lying in the spectral flow orbit satisfying

$$v + K(a + \tilde{a}) = N, \quad v = 0, \dots, N-1, \quad a, \tilde{a} \in \frac{1}{2} + \mathbb{Z}_N, \quad (4.20)$$

in the NS-NS sector of (4.2). The expected lightest state is obtained by setting  $v = N - K$ ,  $a = \tilde{a} = \frac{1}{2}$  (chiral primary). This is wrong GSO projected due to the thermal phase factor  $\epsilon(\sigma, *)$  (3.13), and has the conformal weights  $h = \tilde{h} = \frac{1}{2}$ <sup>5</sup>. We also note that all the states appearing in the R-R, NS-R sectors with thermal winding have conformal weights greater than  $\frac{k}{4}$ , which again corresponds to the Hawking temperature.

To summarize, we have found a massless state with non-vanishing thermal winding in the NS-NS sector. This means that the physical excitations lying in  $Z^{(\text{fin})}$  behave as if we were at the Hagedorn temperature, which is much higher than the Hawking temperature.

One could interpret this light excitation, from a geometrical viewpoint, as a contribution from the string wound around a small circle very close to the tip of cigar. However, it seems interesting to ask why we *exactly* observe the Hagedorn temperature. We would like to comment on a similarity to the effective Hagedorn behavior observed in the spectrum of closed string emission [21] from the ‘rolling D-branes’ [22]. In cases of [21], the effective high temperature behavior would be interpreted as those caused by a very short ‘thermal’ open string attached to points close to the tip of the ‘hairpin shaped D-brane’ [23], which is Wick rotated to the rolling D-brane. However, similarly to the present situation, it is not so obvious why the Hagedorn temperature emerges precisely.

We finally point out that the ‘modular completed’ partition function  $Z^{(\text{dis})}$  [8] given in (4.4), (4.5) independently shows this effective Hagedorn behavior, since it is modular invariant and contains the winding massless state considered above. This is actually the ‘minimum part’ that has this property. Namely, the remaining function  $Z^{(\text{reg})} - Z^{(\text{dis})}$  normally behaves in the UV-region consistent with the Hawking temperature (2.8).

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<sup>5</sup>It would be worth pointing that massless winding state presented here and the no winding one (4.16) are identified with the two Liouville potentials (screening charges) ‘ $S^\pm$ ’ in the dual picture of  $\mathcal{N} = 2$  Liouville theory [16, 11, 17].



## 4.4 Spectrum in the $\mathbb{Z}_M$ -orbifold

Because of the translational invariance along the Euclidean time direction  $t_E$ , one may consider the  $\mathbb{Z}_M$ -orbifold of  $SL(2; \mathbb{R})/U(1)$ -sector. By making this orbifolding, one will gain a deficit angle at the tip and the geometry gets singular, which would affect the analysis given above. Motivated by this expectation, let us explore the spectrum of light excitations in that background. We assume  $N = ML$ ,  $M, L \in \mathbb{Z}_{>0}$ ,  $M \geq 2$ <sup>6</sup>, and the regularized partition function (3.5) is replaced with

$$\begin{aligned} Z_{[\sigma_L, \sigma_R]}^{\text{orb}, (\text{reg})}(\tau; \varepsilon) &= \frac{k}{M} \sum_{m_i \in \mathbb{Z}} \int_{\varepsilon}^{1-\varepsilon} ds_1 \int_0^1 ds_2 \epsilon(\sigma_L; m_1, m_2) \epsilon(\sigma_R; m_1, m_2) \\ &\quad \times f_{[\sigma_L]}(\tau, s_1\tau + s_2) [f_{[\sigma_R]}(\tau, s_1\tau + s_2)]^* e^{-\frac{\pi k}{\tau_2} |(s_1 + \frac{m_1}{M})\tau + (s_2 + \frac{m_2}{M})|^2}. \end{aligned} \quad (4.21)$$

We can likewise analyze this partition function and decompose it in the similar manner to (4.1), (4.4) and (4.8). As is expected, it is found that the asymptotic part is precisely interpreted as the free superstring gas on the background

$$S_{\beta_{\text{Hw}}/M}^1 \times \mathbb{R}_\phi \times SU(2)_k \times T^5,$$

with the temperature  $MT_{\text{Hw}}$ .

Let us turn our focus to the discrete part. By extending the analysis presented in [9] so that the spin structures are included, one can reach the following (modular completed) discrete partition function;

$$\begin{aligned} Z_{[\sigma_L, \sigma_R]}^{\text{orb}, (\text{dis})}(\tau) &= \sum_{v=0}^{N-1} \sum_{\substack{(a, \tilde{a}) \in \mathcal{R}(\sigma_L, \sigma_R, v; M) \\ a \in \mathbb{Z}_N + \frac{s(\sigma_L)-1}{2}, \tilde{a} \in \mathbb{Z}_N + \frac{s(\sigma_R)-1}{2}}} \epsilon(\sigma_L; v, a, \tilde{a}) \epsilon(\sigma_R; v, a, \tilde{a}) \\ &\quad \times \widehat{\chi}_{\text{dis}}^{(\sigma_L)}(v, a; \tau) \overline{\widehat{\chi}_{\text{dis}}^{(\sigma_R)}(v, \tilde{a}; \tau)} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^3 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^3} \\ &= \sum_{v=0}^{N-1} \sum_{(\nu, \tilde{\nu}) \in \mathcal{R}(\sigma_L, \sigma_R, v; M)} \epsilon(\sigma_L; v, a, \tilde{a}) \epsilon(\sigma_R; v, a, \tilde{a}) \\ &\quad \times \widehat{\text{ch}}_{\text{dis}}^{(\sigma_L)}\left(\frac{v}{K}, \nu; \tau\right) \overline{\widehat{\text{ch}}_{\text{dis}}^{(\sigma_R)}\left(\frac{v}{K}, \tilde{\nu}; \tau\right)} \left( \frac{\theta_{[\sigma_L]}}{\eta} \right)^3 \overline{\left( \frac{\theta_{[\sigma_R]}}{\eta} \right)^3}, \end{aligned} \quad (4.22)$$

---

<sup>6</sup>It is always possible for an arbitrary positive integer  $M$ , since we do not assume  $N$  and  $K$  are coprime.

where the range of summation  $\mathcal{R}(\sigma_L, \sigma_R, v; M)$  has been defined as

$$\mathcal{R}(\sigma_L, \sigma_R, v; M) := \left\{ (\nu, \tilde{\nu}) \mid \nu \in \mathbb{Z} + \frac{s(\sigma_L) - 1}{2}, \tilde{\nu} \in \mathbb{Z} + \frac{s(\sigma_R) - 1}{2}, \right. \\ \left. v + K(\nu + \tilde{\nu}) \in L\mathbb{Z}, \nu - \tilde{\nu} \in M \left( \mathbb{Z} + \frac{s(\sigma_L)}{2} + \frac{s(\sigma_R)}{2} \right) \right\}. \quad (4.23)$$

Now, what spectrum will (4.22) yield? The analysis becomes slightly complicated, and one has to be again careful about whether the GSO projection act correctly in the NS-sector. The results are summarized as follows;

**(i) no winding spectrum:**

As in the previous analysis, we have a unique massless state and no tachyons in the NS-NS sector, and all the states with other spin structures are found to be massive. This is just given by (4.16), which clearly belongs to the range (4.23) for an arbitrary value of  $M$ . Consequently, we have a universal massless excitation irrespective of the orbifolding. It should be identified with the modulus  $\mu$  (2.2).

**(ii) thermal winding spectrum:**

We note that the thermal winding number  $w \in \mathbb{Z}$  is now identified as

$$v + K(\nu + \tilde{\nu}) = Lw \left( \equiv \frac{N}{M}w \right). \quad (4.24)$$

Taking account of how the correct/wrong GSO projections act, we obtain the unique lightest excitation

$$v = L - K, \quad \nu = \tilde{\nu} = -\frac{1}{2}, \quad (4.25)$$

which belongs to the NS-NS sector with the winding  $w = 1$ . This state is wrong GSO projected and possesses the conformal weights<sup>7</sup>

$$h = \tilde{h} = \frac{L}{2N} = \frac{1}{2M} < \frac{1}{2}. \quad (4.26)$$

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<sup>7</sup>It might be worthwhile to comment on the following fact: contrary to the unorbifolded case, the R-R vacua exist also in the winding sectors with  $w = 1, \dots, M - 1$ . They have the equal conformal weight  $h = \tilde{h} = \frac{1}{2} + \frac{1}{4k}$ , which are massive but smaller than the weight for the thermal tachyon in the asymptotic region  $\frac{1}{M^2} \frac{k}{4}$  as long as  $M$  is sufficiently small. On the other hand, all the R-R winding states in the unorbifolded background have the weight greater than  $\frac{k}{4}$ .

Thus, it is tachyonic. It is also not difficult to show that all the other states that belong to the partition function (4.22) are massive.

In this way, we have now achieved the Hagedorn-like behavior again, but *above* the Hagedorn temperature. Namely, a tachyonic instability is caused by the thermal winding states. Note that  $L \geq K$  has to be satisfied for the thermal tachyon (4.25) to exist. This constraint is equivalent with  $M \leq k$ , which is generic enough since we assumed a large  $k$ .

Emergence of such a tachyonic excitation would reflect the fact that the background contains a singularity. It is also interesting that the conformal weight (4.26) is proportional to  $M^{-1}$ , rather than  $M^{-2}$ , in contrast with the thermal tachyon in the asymptotic sector, which has the conformal weight

$$h = \tilde{h} = \frac{1}{M^2} \cdot \frac{k}{4} \equiv \left( \frac{1}{M} \frac{\beta_{\text{Hw}}}{2\pi} \right)^2.$$

We finally comment on the remaining sector  $Z^{\text{orb}, (\text{fn})} - Z^{\text{orb}, (\text{dis})}$ . Again this sector does not alter the relevant behavior of partition function, although the GSO-projection acts intricately: states with no winding in this sector have conformal weights greater than  $\frac{1}{2} + \frac{1}{4k}$ , while the conformal weights of winding states always satisfy the inequality  $h \geq \frac{1}{M^2} \frac{k}{4}$ .

## 5 Summary and Comments

In this paper we have studied the thermal torus partition function of superstring propagating in the near-horizon region of the near-extremal black NS5-brane background, which is described by the superconformal system (2.4).

Main results are summarized as follows:

- The thermal partition function has been decomposed into two parts;

$$Z^{(\text{reg})}(\tau) = Z^{(\text{asp})}(\tau) + Z^{(\text{fn})}(\tau),$$

which we called, the ‘asymptotic part’ and the ‘finite part’.

- The asymptotic part  $Z^{(\text{asp})}(\tau)$  is contributed from strings freely propagating in the region far from the NS5-branes. This part is written in the same form as the free superstring gas at the Hawking temperature in the linear-dilaton background, as already given in [14]. It includes contributions from free fermions with the correct thermal boundary conditions consistent with the GSO projection, when making the Wick rotation as given in [3].

- The finite part  $Z^{(\text{fin})}(\tau)$  is a novel result of this paper. It captures the contribution of strings localized around the ‘tip of cigar’, which includes states not necessarily GSO projected in the usual sense, and thus break the space-time SUSY even in the Lorentzian background. This part only includes two massless states as the lightest state in the NS-NS sector, and all the excitations with other spin structures are found to be massive. One of the massless excitations is identified as the energy excess above the extremality.

Another massless state carries a non-vanishing thermal winding. This fact implies that the Hagedorn-like behavior is effectively observed, although the Hawking temperature is much lower than the Hagedorn one. The ‘minimum part’ that shows this effective Hagedorn behavior is identified as the discrete part  $Z^{(\text{dis})}(\tau)$  (4.5), written in terms of the modular completions (C.28).

The  $\mathbb{Z}_M$ -orbifold along the Euclidean time direction has been similarly analyzed. The NS-NS modulus with no winding universally exists irrespective of the orbifolding. The effective Hagedorn behavior still happens. However, we this time observe a temperature higher than the Hagedorn one (2.10): the lightest thermal winding state becomes tachyonic, which would reflect the singularity of background.

We add several comments:

1. Based on the standard argument of thermal string theory [2, 3], the thermal partition function is reinterpreted as the free energy of superstring gas by setting  $m_1(\equiv w) = 0$  and replacing the fundamental region  $\mathcal{F}$  with the strip  $\mathcal{S}$  (4.14) [15]. However, one should *not* suppose that the massless excitation with no winding we found belongs to the physical Hilbert space of single string, after going back to the Lorentzian background. When translating the torus partition function into the free energy of multi-string systems, each sector with the temporal winding  $m_2$  is identified as the contribution from the physical Hilbert space of  $m_2$ -string states, and recall that the temporal winding  $m_2$  has been dualized into the KK momentum  $n$  in our analysis. Therefore, the massless excitation we found would rather correspond to a *collective* excitation of superstring gas.
2. It has been believed that the NS5-brane systems should be described by the Little String Theory (LST) [11] after taking a suitable decoupling limit, which is assumed to be a 6-dimensional non-perturbative and non-gravitational string theory. The fundamental superstring on the background (2.7) is interpreted as the holographic dual of the thermal LST. The Hagedorn temperature of LST has been claimed to be equal the Hawking temperature  $T_{\text{Hw}}$  (2.8) [18] based on the argument of microscopic origin of near-extremal

black-hole entropy. Along this line, the Hagedorn behavior of LST has been investigated in [19, 20, 14]. Especially, in the paper [14], the perturbative region of fundamental superstring (that is, (2.6) is assumed) has been claimed to be dual to the LST slightly above  $T_{\text{Hg}}^{(\text{LST})} (\equiv T_{\text{Hw}})$ , and thus a thermal instability should emerge. The effective Hagedorn behavior observed in  $Z^{(\text{fn})}(\tau)$  seems to be consistent with this claim.

3. As we discussed in section 4, the most relevant part for the effective Hagedorn behavior is the discrete partition function  $Z^{(\text{dis})}$  (4.5) that is modular invariant. Recall that the modular completion  $\hat{\chi}_{\text{dis}}^{(\sigma)}$  (C.28) roughly has the structure

$$\hat{\chi}_{\text{dis}}^{(\sigma)} = \chi_{\text{dis}}^{(\sigma)} + \sum [\text{non-holomorphic, massive}]. \quad (5.1)$$

Because of the modular property of  $\hat{\chi}_{\text{dis}}^{(\sigma)}$  (C.31) (see also (C.19)) and the fact that the modular S-transformation exchanges the UV and IR regions in the torus moduli space, one can conclude that the second term in (5.1) should dominantly contribute in the UV region. While this term just provides a small correction to the discrete character  $\chi_{\text{dis}}^{(\sigma)}$  in the IR region, *it dominates under the UV limit  $\tau \rightarrow +0$* , since it includes much more terms in its  $q$ -expansion than those of  $\chi_{\text{dis}}^{(\sigma)}$ . The accumulation of high energy excitations appearing there could affect the UV behavior of the partition function, leading to a thermal behavior typical at the Hagedorn temperature.

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## Appendix A: Conventions for Theta Functions

We assume  $\tau \equiv \tau_1 + i\tau_2$ ,  $\tau_2 > 0$  and set  $q := e^{2\pi i\tau}$ ,  $y := e^{2\pi iz}$ ;

$$\begin{aligned}
\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \\
\theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \\
\theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-1/2})(1 + y^{-1}q^{m-1/2}), \\
\theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1/2})(1 - y^{-1}q^{m-1/2}).
\end{aligned} \tag{A.1}$$

$$\Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}. \tag{A.2}$$

We also set

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.3}$$

The spectral flow properties of theta functions are summarized as follows ( $m, n, a \in \mathbb{Z}$ ,  $k \in \mathbb{Z}_{>0}$ );

$$\begin{aligned}
\theta_1(\tau, z + m\tau + n) &= (-1)^{m+n} q^{-\frac{m^2}{2}} y^{-m} \theta_1(\tau, z), \\
\theta_2(\tau, z + m\tau + n) &= (-1)^n q^{-\frac{m^2}{2}} y^{-m} \theta_2(\tau, z), \\
\theta_3(\tau, z + m\tau + n) &= q^{-\frac{m^2}{2}} y^{-m} \theta_3(\tau, z), \\
\theta_4(\tau, z + m\tau + n) &= (-1)^m q^{-\frac{m^2}{2}} y^{-m} \theta_4(\tau, z), \\
\Theta_{a,k}(\tau, 2(z + m\tau + n)) &= q^{-km^2} y^{-2km} \Theta_{a+2km,k}(\tau, 2z).
\end{aligned} \tag{A.4}$$

We also use the following identities in the main text;

$$\begin{aligned}
\frac{\theta_3(\tau, u)}{\theta_1(\tau, u)} &= \frac{\theta_3(\tau, 0)}{i\eta(\tau)^3} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i u(n+\frac{1}{2})}}{1 + q^{n+\frac{1}{2}}}, \\
\frac{\theta_4(\tau, u)}{\theta_1(\tau, u)} &= \frac{\theta_4(\tau, 0)}{i\eta(\tau)^3} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i u(n+\frac{1}{2})}}{1 - q^{n+\frac{1}{2}}}, \\
\frac{\theta_2(\tau, u)}{\theta_1(\tau, u)} &= \frac{\theta_2(\tau, 0)}{i\eta(\tau)^3} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i u n}}{1 + q^n}.
\end{aligned} \tag{A.5}$$

They hold for  $u \equiv s_1\tau + s_2$ ,  $0 < s_1 < 1$ , which are proven by using the identity given *e.g.* in [8].

## Appendix B: Irreducible and Extended Characters and their Modular Completions with General Spin Structures

In this appendix we summarize the definitions as well as useful formulas for the (extended) characters and their modular completions of the  $\mathcal{N} = 2$  superconformal algebra with  $\hat{c} (\equiv \frac{c}{3}) = 1 + \frac{2}{k}$ . We shall include here the formulas with general spin structures extending those given in [8, 9]. We assume  $k = N/K$ ,  $N, K \in \mathbb{Z}_{>0}$  (not assumed to be coprime), when considering the extended characters.

To express spin structures concisely, we shall use the notation;

$$\theta_{[\sigma]}(\tau, z) := \theta_3(\tau, z), \theta_4(\tau, z), \theta_2(\tau, z), -i\theta_1(\tau, z), \quad (\text{C.1})$$

for  $\sigma = \text{NS}, \widetilde{\text{NS}}, \text{R}, \widetilde{\text{R}}$  respectively. We also set

$$s(\sigma) := \begin{cases} 0 & \sigma = \text{NS}, \widetilde{\text{NS}}, \\ 1 & \sigma = \text{R}, \widetilde{\text{R}} \end{cases} \quad t(\sigma) := \begin{cases} 0 & \sigma = \text{NS}, \text{R}, \\ 1 & \sigma = \widetilde{\text{NS}}, \widetilde{\text{R}} \end{cases}$$

and

$$\kappa(\sigma) := \begin{cases} 1 & \sigma = \text{NS}, \widetilde{\text{NS}}, \text{R} \\ -i & \sigma = \widetilde{\text{R}} \end{cases}$$

It is also convenient to introduce the notations;

$$S \cdot \text{NS} = \text{NS}, \quad S \cdot \widetilde{\text{NS}} = \text{R}, \quad S \cdot \text{R} = \widetilde{\text{NS}}, \quad S \cdot \widetilde{\text{R}} = \widetilde{\text{R}}, \quad (\text{C.2})$$

$$T \cdot \text{NS} = \widetilde{\text{NS}}, \quad T \cdot \widetilde{\text{NS}} = \text{NS}, \quad T \cdot \text{R} = \text{R}, \quad T \cdot \widetilde{\text{R}} = \widetilde{\text{R}}, \quad (\text{C.3})$$

to write down the modular transformation formulas.

### Continuous (non-BPS) Characters:

$$\text{ch}^{(\sigma)}(P, \mu; \tau, z) := q^{\frac{P^2 + \mu^2}{4k}} y^{\frac{\mu}{k}} \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}, \quad (\text{C.4})$$

which is associated to the irrep. with the following conformal weight  $h$  and  $U(1)$ -charge  $Q$ ;

$$\begin{aligned} h &= \frac{P^2 + \mu^2}{4k} + \frac{1}{4k}, & Q &= \frac{\mu}{k}, & (\text{for } \sigma = \text{NS}, \widetilde{\text{NS}}) \\ h &= \frac{P^2 + \mu^2}{4k} + \frac{\hat{c}}{8}, & Q &= \frac{\mu}{k} \pm \frac{1}{2}, & (\text{doubly degenerated}), \quad (\text{for } \sigma = \text{R}, \widetilde{\text{R}}) \end{aligned} \quad (\text{C.5})$$

The modular transformation formulas and the spectral flow property are given by

$$\text{ch}^{(\sigma)}\left(P, \mu; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \frac{1}{2k} \int_{-\infty}^{\infty} dP' \int_{-\infty}^{\infty} d\mu' e^{2\pi i \frac{PP' - \mu\mu'}{2k}} \text{ch}^{(S \cdot \sigma)}(P', \mu'; \tau, z). \quad (\text{C.6})$$

$$\text{ch}^{(\sigma)}(P, \mu; \tau + 1, z) = e^{2\pi i \left( \frac{P^2 + \mu^2}{4k} + \frac{s(\sigma) - 1}{8} \right)} \text{ch}^{(T \cdot \sigma)}(P, \mu; \tau, z), \quad (\text{C.7})$$

$$\text{ch}^{(\sigma)}(P, \mu; \tau, z + n_1\tau + n_2) = (-1)^{t(\sigma)n_1 + s(\sigma)n_2} e^{2\pi i \frac{\mu}{k} n_2} q^{-\frac{\hat{c}}{2} n_1^2} y^{-\hat{c} n_1} \text{ch}^{(\sigma)}(P, \mu + 2n_1; \tau, z), \quad (\forall n_i \in \mathbb{Z}). \quad (\text{C.8})$$

### Discrete (BPS) Characters [26, 27]:

$$\text{ch}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau, z) := \frac{(yq^\nu)^{\frac{\lambda}{k}}}{1 + (-1)^{t(\sigma)} yq^\nu} y^{\frac{2\nu}{k}} q^{\frac{\nu^2}{k}} \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}, \quad \left( 0 \leq \lambda \leq k, \quad \nu \in \mathbb{Z} + \frac{s(\sigma) - 1}{2} \right). \quad (\text{C.9})$$

- For  $\sigma = \text{NS}, \widetilde{\text{NS}}$  cases, this character is associated to the  $(\nu - \frac{1}{2})$ -th spectral flow of discrete irrep. generated by the chiral primary with

$$h = \frac{Q}{2} = \frac{\lambda + 1}{2k}, \quad (0 \leq \lambda \leq k). \quad (\text{C.10})$$

- For  $\sigma = \text{R}, \widetilde{\text{R}}$  cases, this character is associated to the  $\nu$ -th spectral flow of discrete irrep. generated by the Ramond vacua with

$$h = \frac{\hat{c}}{8}, \quad Q = \frac{\lambda}{k} - \frac{1}{2}, \quad (0 \leq \lambda \leq k) \quad (\text{C.11})$$

The modular transformation formulas are given as [25]

$$\begin{aligned} \text{ch}_{\text{dis}}^{(\sigma)}\left(\lambda, \nu; -\frac{1}{\tau}, \frac{z}{\tau}\right) &= \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \left[ \frac{i}{k} \int_0^k d\lambda' \sum_{\nu' \in \mathbb{Z} + \frac{t(\sigma) - 1}{2}} e^{2\pi i \frac{\lambda\lambda' - (\lambda + 2\nu)(\lambda' + 2\nu')}{2k}} \text{ch}_{\text{dis}}^{(S \cdot \sigma)}(\lambda', \nu'; \tau, z) \right. \\ &\quad \left. + \frac{1}{2k} \int_{-\infty}^{\infty} d\mu' e^{-2\pi i \frac{(\lambda + 2\nu)\mu'}{2k}} \int_{\mathbb{R}(+i0)} dP' \frac{e^{-2\pi i \frac{\lambda P'}{2k}}}{1 + (-1)^{s(\sigma)} e^{-\pi(P' + i\mu')}} \text{ch}^{(S \cdot \sigma)}(P', \mu'; \tau, z) \right] \end{aligned} \quad (\text{C.12})$$

$$\text{ch}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau + 1, z) = e^{2\pi i \left\{ \frac{\nu}{k}(\lambda + \nu) + \frac{s(\sigma) - 1}{8} \right\}} \text{ch}_{\text{dis}}^{(T \cdot \sigma)}(\lambda, \nu; \tau, z). \quad (\text{C.13})$$



### Extended Continuous (non-BPS) Characters [25, 12]:

Set  $k = N/K$ ,  $(N, K \in \mathbb{Z}_{>0})$ .

$$\begin{aligned}\chi_{\text{con}}^{(\sigma)}(p, m; \tau, z) &:= \sum_{n \in N\mathbb{Z}} (-1)^{nt(\sigma)} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n} \text{ch}^{(\sigma)}\left(\frac{p}{K}, \frac{m}{K}; \tau, z + n\tau\right) \\ &= q^{\frac{p^2}{4NK}} \Theta_{m, NK}\left(\tau, \frac{2z}{N}\right) \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}.\end{aligned}\quad (\text{C.14})$$

$h = \frac{p^2+m^2+K^2}{4NK} + \frac{s(\sigma)}{8}$ ,  $Q = \frac{m}{N} \pm \frac{s(\sigma)}{2}$  ( $p \geq 0$ ,  $m \in \mathbb{Z}_{2NK}$ , doubly degenerated for  $\sigma = \text{R}, \tilde{\text{R}}$ ),  
The modular and spectral flow properties are simply written as

$$\chi_{\text{con}}^{(\sigma)}\left(p, m; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \frac{1}{2NK} \int_{-\infty}^{\infty} dp' \sum_{m' \in \mathbb{Z}_{2NK}} e^{2\pi i \frac{pp' - mm'}{2NK}} \chi_{\text{con}}^{(S \cdot \sigma)}(p', m'; \tau, z). \quad (\text{C.15})$$

$$\chi_{\text{con}}^{(\sigma)}(p, m; \tau + 1, z) = e^{2\pi i \frac{p^2+m^2}{4NK} + \frac{s(\sigma)-1}{8}} \chi_{\text{con}}^{(T \cdot \sigma)}(p, m; \tau, z), \quad (\text{C.16})$$

$$\chi_{\text{con}}^{(\sigma)}(p, m; \tau, z + n_1\tau + n_2) = (-1)^{t(\sigma)n_1 + s(\sigma)n_2} e^{2\pi i \frac{m}{N}n_2} q^{-\frac{\hat{c}}{2}n_1^2} y^{-\hat{c}n_1} \chi_{\text{con}}^{(\sigma)}(p, m + 2Kn_1; \tau, z), \quad (\forall n_i \in \mathbb{Z}). \quad (\text{C.17})$$

### Extended Discrete (BPS) Characters [25, 12, 24]:

$$\begin{aligned}\chi_{\text{dis}}^{(\sigma)}(v, a; \tau, z) &:= \sum_{n \in N\mathbb{Z}} (-1)^{nt(\sigma)} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n} \text{ch}_{\text{dis}}^{(\sigma)}\left(\frac{v}{K}, a; \tau, z + n\tau\right) \\ &= \sum_{n \in \mathbb{Z}} \text{ch}_{\text{dis}}^{(\sigma)}\left(\frac{v}{K}, a + Nn; \tau, z\right) \\ &= \sum_{n \in \mathbb{Z}} \frac{(yq^{Nn+a})^{\frac{v}{N}}}{1 + (-1)^{t(\sigma)} yq^{Nn+a}} y^{2K(n+\frac{a}{N})} q^{NK(n+\frac{a}{N})^2} \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}.\end{aligned}\quad (\text{C.18})$$

$$(v = 0, 1, \dots, N-1, \quad a \in \mathbb{Z}_N + \frac{s(\sigma)-1}{2}).$$

- $\sigma = \text{NS}, \tilde{\text{NS}}$  :

This corresponds to the sum of the chiral primary representation with  $h = \frac{1}{2}Q = \frac{v+K}{2N}$ , ( $v = 0, 1, \dots, N-1$ ) over the spectral flows with flow momenta  $m$  taken to be  $m \in a - \frac{1}{2} + N\mathbb{Z}$ , ( $a \in \mathbb{Z}_N + \frac{1}{2}$ ). Especially, in the case of  $a = -\frac{1}{2} (\equiv N - \frac{1}{2})$ , the corresponding spectral flow orbit is the one generated by the anti-chiral primary with  $h = -\frac{1}{2}Q = \frac{N-v+K}{2N}$ .

- $\sigma = \text{R}, \tilde{\text{R}}$  :

This corresponds to the sum of the Ramond vacuum representation with  $h = \frac{\hat{c}}{8}$ ,  $Q = \frac{v}{N} - \frac{1}{2}$  ( $v = 0, 1, \dots, N-1$ ) over spectral flow with flow momentum  $m$  taken to be  $m \in a + N\mathbb{Z}$ .

The modular transformation formula can be expressed as [25, 12, 24];

$$\begin{aligned} \chi_{\text{dis}}^{(\sigma)} \left( v, a; -\frac{1}{\tau}, \frac{z}{\tau} \right) &= \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \left[ \sum_{v'=0}^{N-1} \sum_{a' \in \mathbb{Z}_N + \frac{t(\sigma)-1}{2}} \frac{i}{N} e^{2\pi i \frac{vv' - (v+2Ka)(v'+2Ka')}{2NK}} \chi_{\text{dis}}^{(S \cdot \sigma)}(v', a'; \tau, z) \right. \\ &\quad \left. + \frac{1}{2NK} \sum_{m' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{(v+2Ka)m'}{2NK}} \int_{\mathbb{R}+i0} dp' \frac{e^{-2\pi \frac{vp'}{2NK}}}{1 + (-1)^{s(\sigma)} e^{-\pi \frac{p'+im'}{K}}} \chi_{\text{con}}^{(S \cdot \sigma)}(p', m'; \tau, z) \right] \end{aligned} \quad (\text{C.19})$$

$$\chi_{\text{dis}}^{(\sigma)}(v, a; \tau + 1, z) = e^{2\pi i \frac{a}{N}(v+Ka)} \chi_{\text{dis}}^{(T \cdot \sigma)}(v, a; \tau, z), \quad (\text{C.20})$$

The spectral flow property is also expressed as [24]

$$\chi_{\text{dis}}^{(\sigma)}(v, a; \tau, z + n_1\tau + n_2) = (-1)^{t(\sigma)n_1 + s(\sigma)n_2} e^{2\pi i \frac{v+2Ka}{N}n_2} q^{-\frac{\hat{c}}{2}n_1^2} y^{-\hat{c}n_1} \chi_{\text{dis}}^{(\sigma)}(v, a + n_1; \tau, z), \quad (\forall n_i \in \mathbb{Z}), \quad (\text{C.21})$$

### Modular Completion of the Irreducible Discrete Character [9]:

$$\begin{aligned} \widehat{\text{ch}}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau, z) &:= \text{ch}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau, z) \\ &\quad - \frac{1}{2} \sum_{r \in \mathbb{Z}} (-1)^{r(t(\sigma)-1)} \text{sgn}(r+0) \text{Erfc} \left( \sqrt{\frac{\pi \tau_2}{k}} |\lambda + kr| \right) q^{\frac{\nu^2}{k} + \frac{\nu}{k}(\lambda + kr)} y^{\frac{1}{k}(\lambda + kr + 2\nu)} \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3} \\ &= \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3} y^{\frac{2\nu}{k}} q^{\frac{\nu^2}{k}} \left[ \frac{(yq^\nu)^{\frac{\lambda}{k}}}{1 + (-1)^{t(\sigma)} yq^\nu} \right. \\ &\quad \left. - \frac{1}{2} \sum_{r \in \mathbb{Z}} (-1)^{r(t(\sigma)-1)} \text{sgn}(r+0) \text{Erfc} \left( \sqrt{\frac{\pi \tau_2}{k}} |\lambda + kr| \right) (yq^\nu)^{\frac{\lambda + kr}{k}} \right] \\ &= \frac{i\theta_{[\sigma]}(\tau, z)}{2\pi\eta(\tau)^3} \frac{y^{\frac{2\nu}{k}} q^{\frac{\nu^2}{k}}}{1 + (-1)^{t(\sigma)} yq^\nu} \left\{ \int_{\mathbb{R}+i(k-0)} dp + (-1)^{t(\sigma)} \int_{\mathbb{R}-i0} dp (yq^\nu) \right\} \\ &\quad \times \sum_{r \in \mathbb{Z}} (-1)^{r(t(\sigma)-1)} \frac{e^{-\pi \tau_2 \frac{p^2 + (\lambda + kr)^2}{k}} (yq^\nu)^{\frac{\lambda + kr}{k}}}{p - i(\lambda + kr)}, \\ &\quad (0 \leq \lambda \leq k, \quad \nu \in \mathbb{Z} + \frac{s(\sigma) - 1}{2}). \end{aligned} \quad (\text{C.22})$$

In the non-holomorphic terms in (C.22),  $\text{Erfc}(\ast)$  denotes the error-function defined by

$$\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad (\equiv 1 - \text{Erf}(x)). \quad (\text{C.23})$$

The equality in the last line of (C.22) is derived from the integral formula;

$$\frac{1}{i\pi} \int_{\mathbb{R} \mp i0} dp \frac{e^{-\alpha(p^2 + \nu^2)}}{p - i\nu} = \text{sgn}(\nu \pm 0) \text{Erfc}(\sqrt{\alpha}|\nu|), \quad (\nu \in \mathbb{R}, \quad \alpha > 0), \quad (\text{C.24})$$

and by using a simple contour deformation technique.

Now, the modular S-transformation formula is written as

$$\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}\left(\lambda, \nu; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \frac{i}{k} \int_0^k d\lambda' \sum_{\nu' \in \mathbb{Z} + \frac{t(\sigma)-1}{2}} e^{2\pi i \frac{\lambda\lambda' - (\lambda+2\nu)(\lambda'+2\nu')}{2k}} \widehat{\text{ch}}_{\text{dis}}^{(S \cdot \sigma)}(\lambda', \nu'; \tau, z) \quad (\text{C.25})$$

Namely, the continuous term appearing in the R.H.S of (C.12) drops off by taking the modular completion, and the S-transformation formula gets closed within  $\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}$ .

On the other hand, the T-transformation and spectral flow property are preserved by taking the completion;

$$\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau + 1, z) = e^{2\pi i \left\{ \frac{\nu}{k}(\lambda + \nu) + \frac{s(\sigma)-1}{8} \right\}} \widehat{\text{ch}}_{\text{dis}}^{(T \cdot \sigma)}(\lambda, n; \tau, z), \quad (\text{C.26})$$

$$\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}(\lambda, \nu; \tau, z + n_1\tau + n_2) = (-1)^{t(\sigma)n_1 + s(\sigma)n_2} e^{2\pi i \frac{\lambda+2\nu}{k} n_2} q^{-\frac{\hat{c}}{2} n_1^2} y^{-\hat{c} n_1} \widehat{\text{ch}}_{\text{dis}}^{(\sigma)}(\lambda, \nu + n_1; \tau, z), \quad (\forall n_i \in \mathbb{Z}). \quad (\text{C.27})$$

### Modular Completion of the Extended Discrete Characters [8]:

The modular completion of the discrete character  $\chi_{\text{dis}}^{(\sigma)}$  is defined as the spectral flow sum of  $\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}$  (C.22) in the similar manner to (C.18);

$$\begin{aligned} \widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a; \tau, z) &:= \sum_{n \in N\mathbb{Z}} (-1)^{nt(\sigma)} q^{\frac{\hat{c}}{2} n^2} y^{\hat{c} n} \widehat{\text{ch}}_{\text{dis}}^{(\sigma)}\left(\frac{v}{K}, a; \tau, z + n\tau\right) \\ &= \sum_{m \in \mathbb{Z}} \widehat{\text{ch}}_{\text{dis}}^{(\sigma)}\left(\frac{v}{K}, a + Nm; \tau, z\right) \\ &= \chi_{\text{dis}}^{(\sigma)}(v, a; \tau, z) - \frac{1}{2} \sum_{j \in \mathbb{Z}_{2K}} (-1)^{j(t(\sigma)-1)} R_{v+Nj, NK}(\tau) \Theta_{v+Nj+2Ka, NK}\left(\tau, \frac{2z}{N}\right) \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}, \\ &= \frac{i\theta_{[\sigma]}(\tau, z)}{2\pi\eta(\tau)^3} \sum_{n, r \in \mathbb{Z}} \left[ (-1)^{r(t(\sigma)-1)} \frac{(yq^{Nn+a})^{\frac{v+Nr}{N}}}{1 + (-1)^{t(\sigma)} yq^{Nn+a}} y^{2K(n+\frac{a}{N})} q^{NK(n+\frac{a}{N})^2} \right. \\ &\quad \left. \times \left\{ \int_{\mathbb{R}+i(N-0)} dp + (-1)^{t(\sigma)} \int_{\mathbb{R}-i0} dp (yq^{Nn+a}) \right\} \frac{e^{-\pi\tau_2 \frac{p^2+(Nn+a)^2}{NK}}}{p - i(v + Nr)} \right], \\ &\quad (v = 0, 1, \dots, N, \quad a \in \mathbb{Z}_N + \frac{s(\sigma)-1}{2}) \end{aligned} \quad (\text{C.28})$$

where we set

$$\begin{aligned} R_{m,k}(\tau) &:= \sum_{\nu \in m+2k\mathbb{Z}} \text{sgn}(\nu + 0) \text{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}} |\nu|\right) q^{-\frac{\nu^2}{4k}} \\ &= \frac{1}{i\pi} \sum_{\nu \in m+2k\mathbb{Z}} \int_{\mathbb{R}-i0} dp \frac{e^{-\pi\tau_2 \frac{p^2+\nu^2}{k}}}{p - i\nu} q^{-\frac{\nu^2}{4k}}. \end{aligned} \quad (\text{C.29})$$

Conversely the irreducible modular completion  $\widehat{\text{ch}}_{\text{dis}}^{(\sigma)}$  (C.22) is reconstructed from the extended one  $\widehat{\chi}_{\text{dis}}^{(\sigma)}$  (C.28) by taking the ‘continuum limit’ [9];

$$\lim_{\substack{N \rightarrow \infty \\ k \equiv N/K \text{ fixed}}} \widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a; \tau, z) = \widehat{\text{ch}}_{\text{dis}}^{(\sigma)}\left(\lambda \equiv \frac{v}{K}, a; \tau, z\right). \quad (\text{C.30})$$

The modular transformation formulas for  $\widehat{\chi}_{\text{dis}}^{(\sigma)}$  (C.28) are written as

$$\widehat{\chi}_{\text{dis}}^{(\sigma)}\left(v, a; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \kappa(\sigma) e^{i\pi \frac{\hat{c}}{\tau} z^2} \sum_{v'=0}^{N-1} \sum_{a' \in \mathbb{Z}_N + \frac{t(\sigma)-1}{2}} \frac{i}{N} e^{2\pi i \frac{vv' - (v+2Ka)(v'+2Ka')}{2NK}} \widehat{\chi}_{\text{dis}}^{(S \cdot \sigma)}(v', a'; \tau, z), \quad (\text{C.31})$$

$$\widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a; \tau + 1, z) = e^{2\pi i \left\{ \frac{a}{N}(v+Ka) + \frac{s(\sigma)-1}{8} \right\}} \widehat{\chi}_{\text{dis}}^{(T \cdot \sigma)}(v, a; \tau, z). \quad (\text{C.32})$$

Also the spectral flow property is preserved by taking the completion;

$$\widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a; \tau, z + n_1\tau + n_2) = (-1)^{t(\sigma)n_1 + s(\sigma)n_2} e^{2\pi i \frac{v+2Ka}{N} n_2} q^{-\frac{\hat{c}}{2} n_1^2} y^{-\hat{c} n_1} \widehat{\chi}_{\text{dis}}^{(\sigma)}(v, a + n_1; \tau, z), \quad (\forall n_i \in \mathbb{Z}). \quad (\text{C.33})$$

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